Numerical Results of a Flow in a Third Grade Fluid between Two Porous Walls

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Series solution for a steady flow of a third grade fluid between two porous walls is given by the homotopy analysis method (HAM). Comparison with the existing numerical solution is shown. It is found that, unlike the numerical solution, the present series solution holds for all values of the material parameter of a third grade fluid.

Key words: Series Solution; Porous Walls; Comparison; Homotopy Analysis Method.

1. Introduction

During the last few decades the study of non-Newtonian fluids is motivated by their widespread applications in industry. Examples of such fluids are molten plastics, polymers, pulps, foods, and slurries. The non-Newtonian fluids are mainly classified into three categories, namely the differential type, rate type and integral type. One of the simplest subclasses of differential-type fluids is known as the second grade fluid. There is a large body of literature dealing with steady and unsteady flows of a second grade fluid in various situations. Some interesting works on the flows of a second grade fluid that have been reported previously may be mentioned in [1 – 10].

It is a well established fact that second grade fluids exhibit the normal stress effect and do not show the shear-thinning and shear-thickening phenomena [11] which many fluids do. However, third grade fluids [12] are capable of describing such phenomena. Moreover, the equation of motion in a third grade fluid is more complicated than the corresponding equation in a second grade fluid. Very recently, Ariel [13] studied the flow of a third grade fluid bounded by a porous channel. He found that the developed series solution degrades sharply when the material parameter of the third grade fluid is increased. Later Hayat et al. [14] reconsidered the problem of [13] and found the three term homotopy solution valid for all values of the third grade parameter. The present work is an extension of [14] allowing an arbitrary number of terms in the homotopy analysis method (HAM) by different operators to find a new formulation of the HAM.

In the present paper, we will study the flow of a thermodynamic third grade fluid between two porous boundaries. The formulated problem is non-dimensionalized and will be solved using the HAM [15]. This technique is very powerful and has been already employed by many workers [16 – 30] for the series solutions of various problems. Here the convergent series is developed and analyzed.

2. Problem Statement

In this section we briefly introduce the equation for a steady flow of a third grade fluid between two porous walls at $y = 0$ and $y = b$. The x- and y-axes are taken parallel and normal to the channel walls, respectively. There is cross flow of uniform injection of the fluid at the lower wall with velocity V_0 and equal suction at the upper wall. Employing the same line as drawn by Ariel [13] and Hayat et al. [14], the dimensionless problem formulation turned out to be in the form

$$KRU''' + U'' - RU' + TU'^2U'' = -1, \tag{1}$$

$$U(0) = U(1) = 0. \tag{2}$$

In the above equations

$$\eta=\frac{y}{b},\quad U=-\frac{\mu u}{b^2}\left(\frac{\mathrm{d}p}{\mathrm{d}x}\right)^{-1},\quad R=\frac{\rho V_0 b}{\mu},$$
$$K=\frac{\alpha_1}{\rho b^2},\quad T=\frac{6\beta_3 b^2\,(\mathrm{d}p/\mathrm{d}x)^2}{\mu^3},$$

ρ is the density of the fluid, μ is the dynamic viscosity, α_1 and β_3 are dimensional material parameters, p is the pressure, u is the x-component of the velocity, and the prime indicates differentiation with respect to η. Moreover the signs of material parameter are given in [12].

Integration of (1) yields

$$KRU''+U'-RU+\frac{T}{3}U'^3=-\eta+D_1, \tag{3}$$

where D_1 is a constant of integration.

Using

$$U=-\frac{1-\eta}{R}+W, \tag{4}$$

(2) and (3) become

$$KRW''+W'-RW+\frac{T}{3}\left(\frac{1}{R}+W'\right)^3=D_2, \tag{5}$$

$$W(0)=\frac{1}{R},\quad W(1)=0, \tag{6}$$

where

$$D_2=D_1-\left(\frac{1+R}{R}\right).$$

Now letting $\tau=m\eta$, where m is a suitable constant whose value is to be determined later, (5) and (6) reduce to

$$\begin{aligned}&KRm^2W''(\tau)+m\left(1+\frac{T}{R^2}\right)W'(\tau)\\&-RW(\tau)+\frac{T}{R}m^2W'^2(\tau)+\frac{T}{3}m^3W'^3(\tau)=D_3,\end{aligned} \tag{7}$$

where

$$D_3=D_2-\frac{T}{3R^3}$$

and the boundary conditions

$$W(0)=\frac{1}{R},\quad W(m)=0. \tag{8}$$

3. Solution by the HAM

According to (7) and the boundary conditions (8), the solution can be expressed in the form

$$W(\tau)=\sum_{n=0}^{+\infty}c_n\mathrm{e}^{n\tau}, \tag{9}$$

where the c_n $(n=0,1,\ldots)$ are coefficients to be determined. According to the *rule of solution expression* denoted by (9) and the boundary conditions (8), it is natural to choose

$$W_0(\tau)=\frac{\mathrm{e}^m-\mathrm{e}^\tau}{R(\mathrm{e}^m-1)} \tag{10}$$

as the initial approximation to $W(\tau)$.

We define an auxiliary linear operator $\mathcal{L}$ as

$$\begin{aligned}&\mathcal{L}[\phi(\tau;p)]=\\&\left[KRm^2\frac{\partial^2}{\partial\tau^2}+m\left(1+\frac{T}{R^2}\right)\frac{\partial}{\partial\tau}-R\right]\phi(\tau;p)\end{aligned} \tag{11}$$

with the property

$$\mathcal{L}[C\mathrm{e}^\tau]=0, \tag{12}$$

where C is a constant and

$$m=\frac{1}{2KR}\left[-\left(1+\frac{T}{R^2}\right)\pm\sqrt{\left(1+\frac{T}{R^2}\right)^2+4KR}\right].$$

The choice of $\mathcal{L}$ is motivated by (9), choosing

$$m=\frac{1}{2KR}\left[-\left(1+\frac{T}{R^2}\right)+\sqrt{\left(1+\frac{T}{R^2}\right)^2+4KR}\right], \tag{13}$$

and the later requirement that (20) should contain only one non-zero constant.

From (7) we define the non-linear operator

$$\begin{aligned}\mathcal{N}[\phi(\tau;p)]:=&KRm^2\left(\frac{\partial^2\phi}{\partial\tau^2}\right)\\&+m\left(1+\frac{T}{R^2}\right)\left(\frac{\partial\phi}{\partial\tau}\right)-R\phi+\frac{T}{R}m^2\left(\frac{\partial\phi}{\partial\tau}\right)^2\\&+\frac{T}{3}m^3\left(\frac{\partial\phi}{\partial\tau}\right)^3-D_3,\end{aligned} \tag{14}$$

and then construct the homotopy

$$\mathcal{H}[\phi(\tau;p)] = (1-p)\mathcal{L}[\phi(\tau;p) - w_0(\tau)] - \hbar p \mathcal{N}[\phi(\tau;p)], \tag{15}$$

where $\hbar$ is a non-zero auxiliary parameter. Setting $\mathcal{H}[\phi(\tau;p)] = 0$, we have the zero-order deformation equation

$$(1-p)\mathcal{L}[\phi(\tau;p) - w_0(\tau)] = \hbar p \mathcal{N}[\phi(\tau;p)], \tag{16}$$

subject to the boundary conditions

$$\phi(0;p) = \frac{1}{R}, \quad \phi(m;p) = 0, \tag{17}$$

where $p \in [0,1]$ is an embedding parameter. When the parameter p increases from 0 to 1, the solution $\phi(\tau;p)$ varies from $W_0(\tau)$ to $W(\tau)$. If this continuous variation is smooth enough, the Maclaurin's series with respect to p can be constructed for $\phi(\tau;p)$, and further, if this series is convergent at $p = 1$, we have

$$W(\tau) = W_0(\tau) + \sum_{n=1}^{+\infty} W_n(\tau),$$

where

$$W_n(\tau) = \frac{1}{n!}\frac{\partial^n \phi(\tau;p)}{\partial p^n}\Big|_{p=0}.$$

Differentiating (16) and (17) n times with respect to p, then setting $p = 0$, and finally dividing by $n!$, we obtain the nth-order deformation equation

$$\mathcal{L}[W_n(\tau) - \chi_n W_{n-1}(\tau)] = \hbar R_n(\tau) \quad (n = 1,2,3,\ldots), \tag{18}$$

subject to the boundary conditions

$$W_n(0) = 0, \quad W_n(m) = 0, \tag{19}$$

where R_n is defined as

$$R_n(\tau) = KRm^2 W''_{n-1} + m\left(1 + \frac{T}{R^2}\right)W'_{n-1} - RW_{n-1} + \left(\frac{T}{R}m^2\right)\sum_{i=0}^{n-1} W'_i W'_{n-i-1} + \left(\frac{T}{3}m^3\right)\sum_{i=0}^{n-1}\left(W'_{n-i-1}\sum_{j=0}^{i} W'_j W'_{i-j}\right) - D_3(1-\chi_n)$$

with

$$\chi_n = \begin{cases} 0, & n \le 1, \\ 1, & n > 1. \end{cases}$$

Table 1. The variation of the velocity in the middle of the channel, $U(1/2)$, depending the cross-flow Reynolds number, R, the viscoelastic fluid parameter, K, and the third grade parameter, T, at $N = 15$.

R	K	T	Numerical solution [13]	HAM solution [14]	HAM solution	$\hbar$
1	0.1	0	0.112549	⋆	0.112549	[−1.25, −0.05]
		1	0.109334	0.109368	0.109333	−1.25
		2	0.106976	0.106976	0.106767	−1.05
		5	0.100651	0.101969	0.100453	−1.4
	0.2	0	0.105169	⋆	0.105169	[−1.15, −0.05]
		1	0.102101	0.102151	0.102101	−1.2
		2	0.099691	0.099400	0.0997903	−0.9
		5	⋆	⋆	0.0983401	−1.35
	0.5	0	0.090498	⋆	0.0904982	[−2.2, −0.05]
		1	0.087626	0.087673	0.0876259	−1.15
		2	0.085475	0.085471	0.0854748	−1.15
		5	⋆	⋆	0.0817459	−1.1
2	0.1	0	0.091280	⋆	0.0912803	[−1.8, −0.05]
		1	0.089911	0.089912	0.0899108	−1.1
		2	0.088679	0.088692	0.0886788	−1.25
		5	0.085578	0.085708	0.0855739	−1.0
		10	⋆	⋆	0.0813496	−1.3
	0.2	0	0.079166	⋆	0.0791659	[−2.05, −0.05]
		1	0.078179	0.078179	0.0781786	−1.0
		2	0.077282	0.077288	0.0772817	−1.15
		5	0.074993	0.075064	0.0749932	−1.25
		10	⋆	⋆	0.0720381	−1.15
2	0.5	0	0.061230	⋆	0.0612297	[−2.2, −0.05]
		1	0.060643	0.060643	0.0606427	−1.05
		2	0.060099	0.060100	0.0600987	−1.05
		5	0.058673	0.058690	0.0586734	−1.15
		10	⋆	⋆	0.0567829	−1.3
5	0.1	0	0.052205	⋆	0.0522052	[−2.05, −0.05]
		1	0.052089	0.052089	0.0520889	[−1.0, −0.9]
		2	0.051974	0.051974	0.0519741	−1.05
		5	0.051638	0.051638	0.0516376	−1.05
		10	0.051102	0.051102	0.0511024	−1.05
		20	0.050117	0.050119	0.0501174	−1.1
		50	⋆	⋆	0.0476997	−1.35
	0.2	0	0.042011	⋆	0.0420107	[−2.7, −0.05]
		1	0.041948	0.041948	0.041948	−1.1
		2	0.041886	0.041886	0.0418859	[−1.1, −0.95]
		5	0.041703	0.041703	0.0417026	[−1.1, −0.9]
		10	0.041407	0.041407	0.0414073	−1.05
		20	0.040852	0.040852	0.0408516	−1.1
		50	⋆	⋆	0.0394157	−1.2
	0.5	0	0.029777	⋆	0.0297771	[−1.05, −0.9]
		1	0.029750	0.029750	0.0297501	−0.85
		2	0.029723	0.029723	0.0297234	−0.85
		5	0.029644	0.029644	0.0296439	−1.0
		10	0.029514	0.029514	0.0295144	−1.0
		20	0.029266	0.029266	0.0292656	−1.05
		50	⋆	⋆	0.0285919	−1.1

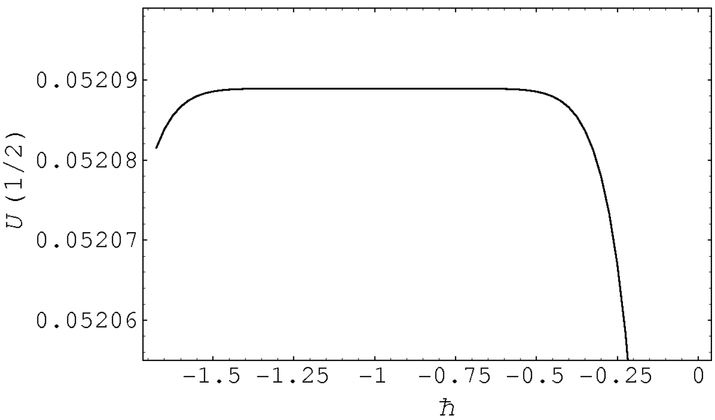

Fig. 1. The velocity in the middle of the channel versus $\hbar$ for the 10th-order approximation with $R = 5$, $K = 0.1$ and $T = 1$.

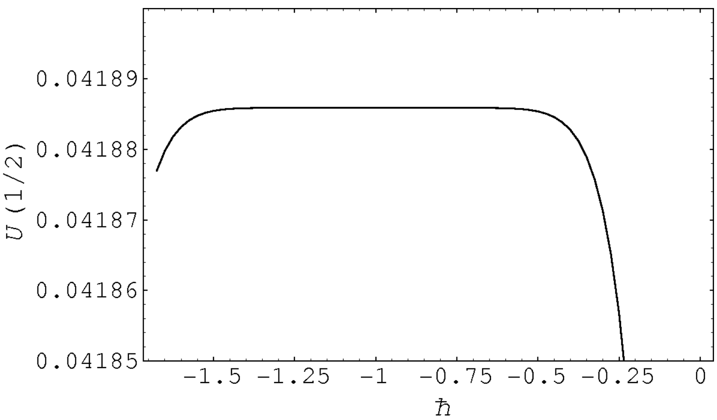

Fig. 2. The velocity in the middle of the channel versus $\hbar$ for the 10th-order approximation with $R = 5$, $K = 0.2$ and $T = 2$.

The general solution of (18) is

$$W_n(\tau) = \hat{W}_n(\tau) + C\mathrm{e}^{\tau}, \tag{20}$$

where C is a constant and $\hat{W}_n(\tau)$ is a particular solution of (18). The unknown C is obtained by the first condition of (19), i. e.,

$$W_n(0) = 0.$$

In this way, we derive $W_n(\tau)$ for $n = 1, 2, 3, \ldots$ successively. At the Nth-order approximation, we have the analytic solution of (7), namely

$$W(\tau) \approx W^{(N)}(\tau) = \sum_{n=0}^{N} W_n(\tau). \tag{21}$$

The auxiliary parameter $\hbar$ can be employed to adjust the convergence region of the series (21) in the homotopy analysis solution. But the obtained solution by (21) has a constant D_1, from (3). Now, we can obtain D_1 by satisfying the second condition of (19), or (8), i. e.,

$$W^{(N)}(m) = 0,$$

for any value of $\hbar$.

4. Numerical Results

By means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series. As pointed out by Liao [15], the appropriate region for $\hbar$ is a horizontal line segment. Our solution series contain the auxiliary parameter $\hbar$. We can choose an appropriate value of $\hbar$ to ensure that the solution series converge. We can investigate the influence of $\hbar$ on the convergence of $U(1/2)$, the velocity in the middle of the channel, by plotting the curve of it versus $\hbar$, as shown in Figs. 1 and 2 for some examples.

Here it is worth mentioning that the HAM solution is valid for all values of the physical parameters R,

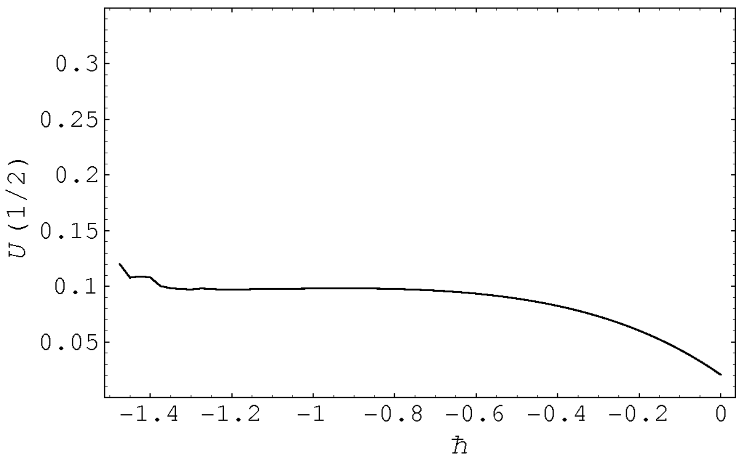

Fig. 3. The velocity in the middle of the channel versus $\hbar$ for the 15th-order approximation with $R = 1$, $K = 0.2$ and $T = 5$.

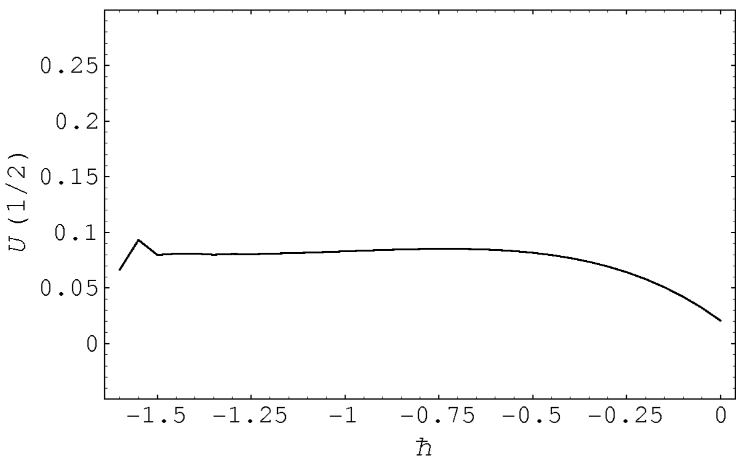

Fig. 4. The velocity in the middle of the channel versus $\hbar$ for the 15th-order approximation with $R = 1$, $K = 0.5$ and $T = 5$.

K, and T. Therefore, it seems reasonable to assume that the HAM solution holds even for those values of the physical parameters for which Ariel [13] had a problem in obtaining the convergence of the series solution for large values of T. In Table 1, $U(1/2)$ is presented for various values of R, K and T, and for selected $\hbar$ with minimum residual error in the middle of the channel, which seem reasonable values according to Figs. 1 and 2, for example. The numerical solutions in the first column of Table 1 were obtained by Ariel [13]. The second column stands for an old formulation of the HAM with three terms, which was obtained in [14]. In Table 1, '$\star$' means that this number was not reported by the mentioned references. The obtained results show that the proper value of $\hbar$ depends considerably on the set of parameters which have been demonstrated in [14] for an old formulation of the HAM.

Figures 3 and 4 show the $\hbar$-curves for $R = 1$, $T = 5$, and $K = 0.2$ and 0.5, respectively. Looking at Table 1, it has to be remarked that Ariel [13] was not able to get the convergence of the solution for the above-mentioned values of the parameters. Thus the HAM offers an attractive alternative for computing the flow of viscoelastic fluids where numerical techniques fail to give the solution for various reasons.

5. Conclusion

We discussed the flow of a nonlinear fluid. An analytical solution of the highly nonlinear problem was constructed. The obtained series solution strongly depends upon the material parameters of the third grade fluid. It was found that the velocity field increases upon increasing the material parameter of a second grade fluid. This study is an extension of [14] allowing an arbitrary number of terms in the HAM by different operators to find a new formulation of the HAM. The HAM provides a convenient way to control the convergence of approximation series; this is a fundamental qualitative difference between the HAM and other methods for finding approximate solutions.

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